ROTATING FRAME ANALYSIS OF RIGID BODY DYNAMICS IN SPACE PHASOR VARIABLES

Shayak Bhattacharjee

Department of Physics,
Indian Institute of Technology Kanpur,
NH-91, Kalyanpur,
Kanpur – 208016,
Uttar Pradesh, India.

*   *   *   *   *

CLASSIFICATION

ABSTRACT

In this work we propose a new method of attacking problems in rigid body rotation. Traditionally, the free symmetric top is solved by applying Euler's equations but that approach is rejected for the heavy top. We here extend the Eulerian analysis to the heavy top, facilitating our calculations using complex (space phasor) variables. Axis transformations are employed to switch between the observable and the body frame components of angular velocity. We cover the cases of free symmetric top, heavy symmetric top, the same with damping in the pivot (damped compensated top), and the aforementioned with the pivot subject to an acceleration in the horizontal plane. In the first two cases, our method provides a simple, direct connection between the initial state and the final behaviour; in the last two cases, it is the asymptotic particular solutions which are of interest. We also include a few suggestions regarding the experimental apparatus for validating our predictions.

*   *   *   *   *

INTRODUCTION

A rather demanding part of any classical mechanics course is the section on rigid body rotation. The counter-intuitive behaviour of tops and gyroscopes, together with the technical difficulties inherent in their analysis, has drawn the attention of teachers and researchers alike. One very standard treatment of this problem is in Goldstein et. al. [1] where they construct the Lagrangian of the rotating top in terms of the Euler angles. Cyclic coordinates reveal first integrals of the motion and these can be used to 'reduce the problem to quadratures'. Solution of the system is another matter as it involves elliptic functions and the profusion of mathematics pushes the underlying physics into the background. Some conclusions can be drawn by observing the signs etc. of various quantities and these lead to a description of the well-known precession and nutational behaviour of the top. This approach is in fact similar to the one followed in Landau and Lifshitz [2].

A second approach has been outlined in the text by Morin [3], which places greater emphasis than Goldstein on physical concepts and intuition. Here, a frame has been considered where the coordinate axes form a principal basis for the top. The frame does not share the top's entire angular velocity though but a part of it. The relation $\mathbf{\Gamma}=\mathrm{d}\mathbf{L}/\mathrm{d}t$ is applied in this basis, and considerable algebraic manipulation is required to obtain the expressions for the time derivatives of the rotating axes. The result is a set of three coupled equations which describe the dynamics of spin, precession and nutation. As Morin has shown, the same equations can be obtained from the Lagrangian, with less effort. The solution of these equations, under certain simplifying assumptions, yields the precessional and nutational patterns. Yet another way of obtaining the same dynamical equations may be found in Peraire et. al. [4] where Euler's equations have been employed in the principal basis used in Morin. Having derived the equations differently, they go on to analyse its consequences in much the same way as Morin.

A novel approach bypassing these equations has been proposed by Schoenhammer [5] where geometrical considerations are used to obtain the precession and nutation. The formal dynamical equations have been bypassed and instead an ingenious geometric construction has been used where components of angular momentum in various bases, including a non-orthogonal system, have been considered. The work is innovative no doubt, but it may often prove difficult to correctly execute the derivation in practice, and apply it to new problems.

Surprisingly all of the previous works use the same formalism for handling the free symmetric top – Euler's equations. The symmetry criterion leads at once to a conserved momentum, and the two other equations possess a symmetry which makes them relatively easy to solve. The body frame angular velocities are obtained from this solution, together with an interpretation in terms of the inertia ellipsoid. The behaviour of the angular velocity as seen from the rotating frame is described; however the approach is not extended to obtain the lab frame angular velocities, which are the quantities of interest in the heavy top.

In this work we start with the free symmetric top and extend the same approach to the heavy top. Further, we introduce complex variables to simplify the calculations and express the results more elegantly. For all kinds of tops, we perform a body frame analysis, using axis transformations as required to switch between the body frame components of angular velocity and their corresponding physically observable components. This general exposition is the focus of Sec. I. The following Sec. II deals with a few concrete applications of the proposed method. To simplify the mathematics, we make two assumptions here – first that the top remains close to the vertical and second that the spin speed about its figure axis is sufficiently high so as not to undergo appreciable changes in the course of precession and nutation. We first consider the free symmetric top and next, the heavy top. We then add on a damping term at the pivot and observe the change in the top's ultimate behaviour. Finally we treat the case where the pivot is given an acceleration in the horizontal plane. We show that the answer obtained from our formalism agrees with physical intuition. We conclude the paper with a summary of the main results and some suggestions regarding experimental verification.

I. EULER'S EQUATIONS IN SPACE PHASOR VARIABLES

We are very familiar with Euler's equations of rigid body rotation i.e.

$$I_1\dot{\omega}_1 + \left(I_3 - I_2\right)\omega_3\omega_2 = \Gamma_1 \quad , \tag{1}$$

and its cyclic permutations but to avoid mistakes we give a brief note of how this equation is derived. It is obtained by transformation from a body fixed frame (the principal basis) i.e. a rotating frame to the lab frame whereby one has to use Coriolis' theorem,

$$\left(\frac{d\mathbf{L}}{dt}\right)_{lab} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega}\times\mathbf{L} \quad , \tag{2}$$

and employ the fact that in the principal basis, $\mathbf{L} = \left(I_1\omega_1, I_2\omega_2, I_3\omega_3\right)$. Application of Euler's equation in a frame sharing part of the angular velocity of the top has been done in [4] as the means to derive the dynamics. In this work, we consider Euler's equations in a frame attached to the top i.e. possessing all of its angular velocity.

We are going to work in three different reference frames (*) – the lab frame (x,y,z), the intermediate frame (a,b,c) and the body fixed frame (d,q,o). All the frames have a common origin – the top's pivot. The x-y-z frame is fixed in space. The axes are orthogonal and gravity acts along the $-z$ direction as is conventional. Making an angle θ with the z axis is the symmetry axis of the top which we denote by the c axis. Considering the plane orthogonal to the c axis, we let the a axis be the line along which this plane intersects the horizontal x-y plane. φ is the angle in the x-y plane between the x and a axes. Orthogonal to a and c is b axis. Finally, the d and q axes are body fixed axes obtained from the a and b axes by a rotation in the a-b plane through an angle ψ. This rotation takes place about the c axis which thus remains invariant but for notational consistency we shall now refer to the same axis as the o axis. Thus, θ becomes the angle of nutation, φ that of precession and ψ that of the top's spin. Descriptions of the axes are meaningless without a diagram and we show several views of the transformations in Figs. 1 and 2.

We write Euler's equations in the d-q-o reference frame, getting

$$I_d\dot{\omega}_d + \left(I_o - I_q\right)\omega_o\omega_q = \Gamma_d \quad , \tag{3a}$$

$$I_q\dot{\omega}_q + \left(I_d - I_o\right)\omega_d\omega_o = \Gamma_q \quad , \tag{3b}$$

$$I_o\dot{\omega}_o + \left(I_q - I_d\right)\omega_q\omega_d = \Gamma_o \quad . \tag{3c}$$

If the top is symmetric I_d will equal I_q and if there be no torque about the o axis, (3c) will at once lead to the conservation of ω_o. We now see that under these conditions there is a symmetry of the remaining two equations which permits the use of the space phasor method.

The space phasor method is used to reduce the order of systems of real differential equations by introducing complex variables, called phasors. For instance, if there be two equations for two real variables, we will form a single phasor variable by adding j (imaginary unit) times the second variable to the first. If the original system can be simplified using this ansatz, then the analysis is worthwhile. The technique finds application in physics in analysis of the Foucault pendulum [6] and in electrical engineering in the study of electrical machinery [7],[8].

Letting $I_d = I_q = I$ and using the conservation of ω_o, we multiply (3b) by j and add it to (3a) getting

$$\frac{d}{dt}\left(\omega_d + j\omega_q\right) + j\frac{I - I_o}{I}\omega_o\left(\omega_d + j\omega_q\right) = \frac{1}{I}\left(\Gamma_d + j\Gamma_q\right) \quad . \tag{4}$$

Quite clearly, a space phasor formulation is possible in terms of the phasor $\omega_d + j\omega_q$ which we denote by $\underline{\boldsymbol{\omega}}$.

(*) A note on the axis nomenclature : x-y-z is of course pretty standard and needs no comment. We have intentionally not used symbols like x'-y'-z' or x_1-x_2-x_3 for the other axes to highlight the fact that they belong to very different reference frames. The notation a-b-c is clear; d-q-o is taken from engineering terminology where d represents "direct" and q "quadrature" meaning "at right angles". o is for "zero-sequence"; a technicality perhaps confusing to the reader who can very reasonably imagine the letter to stand for "orthogonal".
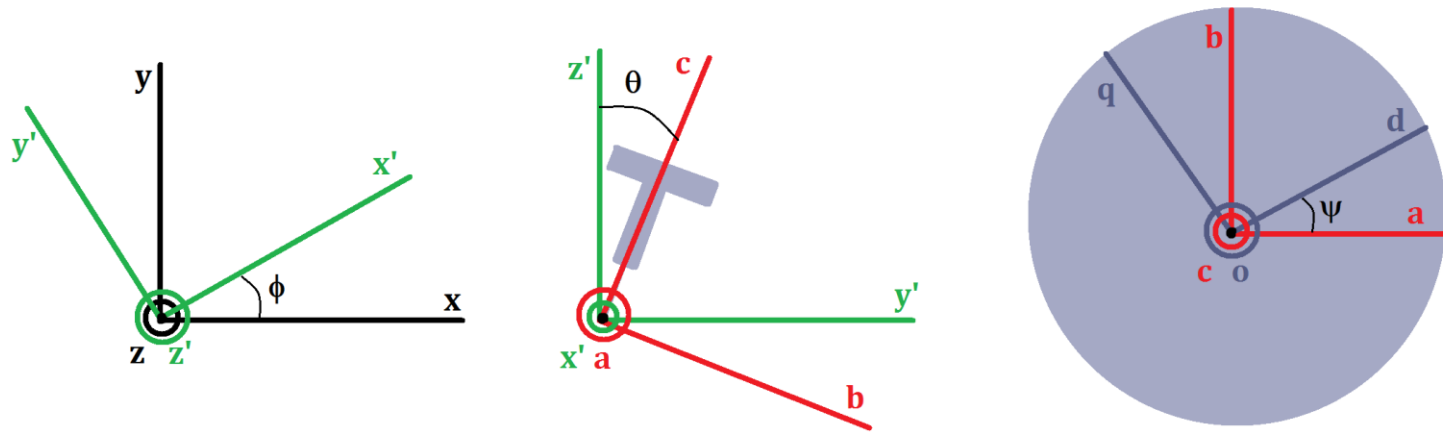
Figure 1 : *Orthographic views of orthogonal transformations. In each view the axis of rotation comes out of the plane of the paper and is shown as a circle in the relevant colour. The x-y-z system is fixed in the lab frame with gravity along the –z direction. The origin is at the top's pivot. The first rotation (left panel) is through angle φ about the z axis to form the x'-y'-z' axes (these axes are useless in the analysis that follows). The next step (middle panel) consists of a rotation through angle θ about the x' axis and the set of axes thus formed is denoted by a-b-c. The axis of symmetry of the top lies along the c axis and the top can be seen in profile view. The final step (right panel) is a rotation through angle ψ about the c axis to get the d-q-o axes. The coloured background highlights the fact that the rotation takes place in the plane of the top's spin. The slate grey colour has been consistently used to indicate the top in this and the next figure; moreover the d q and o axes are shown in a darker shade of the same colour to highlight the fact that they are fixed in the top's frame. We note that a-b-c and d-q-o are both principal axes for the top – however the latter frame carries the entire angular velocity of the top while the former has only a part of it.*
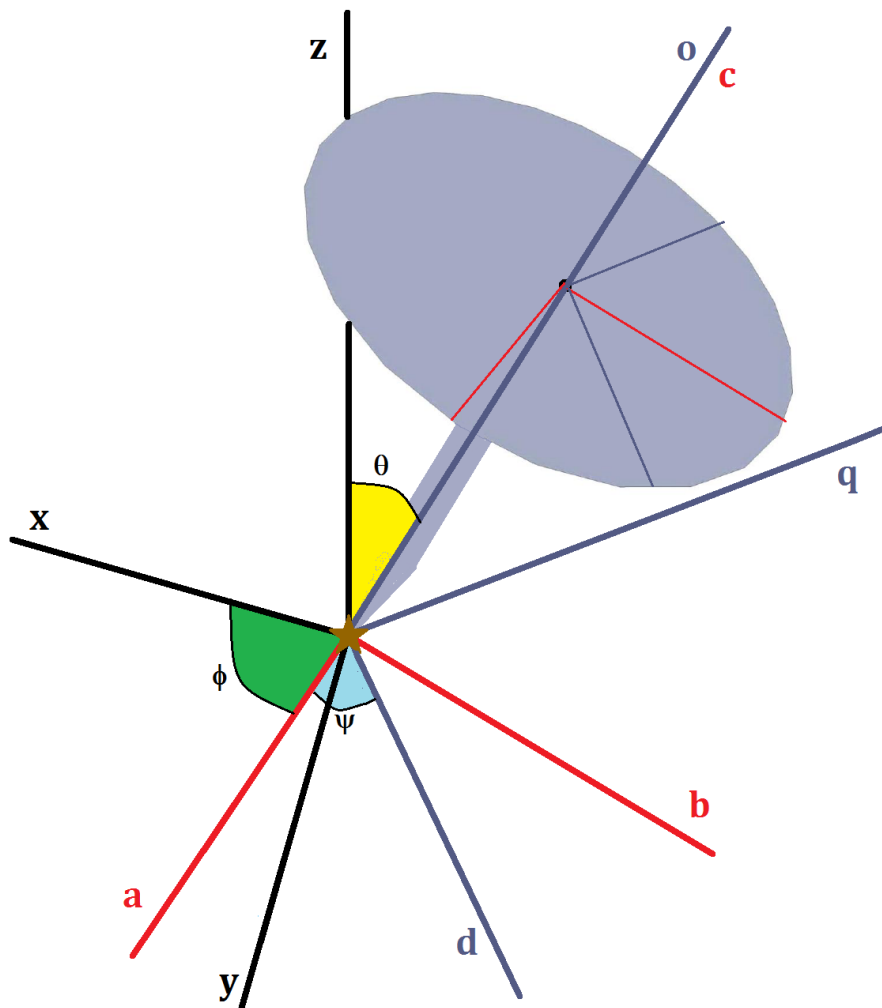

Figure 2 : *Three-dimensional view of the top. The axes and angles mentioned in Fig. 1 can be seen clearly. We should remember that the rotations through φ and ψ are not coplanar.*

Letting $v = \dfrac{I - I_o}{I}\omega_o$ and $\underline{\Gamma} = \Gamma_d + j\Gamma_q$ we obtain the space phasor form of (4) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega} + jv\underline{\omega} = \frac{1}{I}\underline{\Gamma} \quad . \tag{5}$$

This is the equation which must be solved for the dynamics of the body frame angular velocities.

Our next task is to obtain the relations between the observable and body frame components of the angular velocity. For this we need the transformation relations between the various sets of axes. Before presenting the rotation matrices we mention that a formal knowledge of the theory of rotations is absolutely not required for understanding this paper. The transformations have been shown very clearly in Fig. 1 and it is an easy exercise in trigonometry to derive the matrix from the diagram. The transformation from the *a-b-c* to the *d-q-o* basis is simple:

$$\begin{bmatrix} \hat{\mathbf{d}} \\ \hat{\mathbf{q}} \\ \hat{\mathbf{o}} \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{b}} \\ \hat{\mathbf{c}} \end{bmatrix} \quad . \tag{6}$$

We note that θ, φ and ψ are functions of *t*. The complete *x-y-z* to *d-q-o* transformation is a rotation as per the *x*-convention of Goldstein and we quote the formula for it :

$$\begin{bmatrix} \hat{\mathbf{d}} \\ \hat{\mathbf{q}} \\ \hat{\mathbf{o}} \end{bmatrix} = \begin{bmatrix} \cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi & \sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi & \sin\theta\sin\psi \\ -\cos\varphi\sin\psi - \cos\theta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{bmatrix} \quad . \tag{7}$$

Geometrical considerations yield the directions of the various angular velocities : $\dot{\theta}$ is along the *a* axis, $\dot{\varphi}$ along the *z* axis and $\dot{\psi}$ along the *o* axis. Now projecting these onto the *d*, *q* and *o* axes yields the body frame components of the angular velocity as follows.

$$\omega_d = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi \quad , \tag{8a}$$
$$\omega_q = -\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi \quad , \tag{8b}$$
$$\omega_o = \dot{\psi} + \dot{\varphi}\cos\theta \quad . \tag{8c}$$

Since the main equation (5) deals with the space phasor we combine (8a)+j(8b) to get

$$\underline{\omega} = \left(\dot{\theta} + j\dot{\varphi}\sin\theta\right)\underline{\mathbf{1}}\mathrm{e}^{-j\psi} \quad . \tag{9}$$

The $\underline{\mathbf{1}}$ on the right hand side (RHS) of the above refers to the unit phasor 1+j0. It is a technical construct and has been introduced to maintain consistency so that both sides of the equation become phasors. Otherwise the entire coefficient of the exponential would have to be marked as a phasor, which is cumbersome notation. In practice one gets by pretty well without writing the $\underline{\mathbf{1}}$ (I did not while doing these calculations) and in an examination, the candidates should not be penalised for omitting it.

We note that solving (5) and (9) together give us the complete behaviour of the top in terms of nutation, precession and spin. What we have done until this point is simply a mathematical restatement of the formulation of the rigid body problem – we have not obtained a solution ! Also, we have not made any assumptions, approximations or simplifications in this derivation which is thus fully general. The short cuts will come once we start using the method to solve specific problems, which is the subject of the following section.

II. ILLUSTRATIVE EXAMPLES

Before we begin this section we make two assumptions which make the calculations easier. First, we assume that the top always stays close to the vertical so that $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Next, we assume that the top is fast i.e. the spin speed about the o axis is very high. Very high means that as the top precesses and nutates, there will be very little change in the spin speed because of which it can be treated as a constant which we call Ω. Hence we have the relation $\psi = \Omega t$. We note that all the calculations of this section will use these approximations throughout so in future we will not bother to mention them explicitly every time we use them.

1. FREE SYMMETRIC TOP (FST)

This is the easiest case – the torque on the top is simply zero and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega} + jv\underline{\omega} = \mathbf{0} \quad . \tag{10}$$

Solving and using (9) together with the above mentioned approximations we readily get

$$\left(\dot{\theta} + j\theta\dot{\varphi}\right)\underline{\mathbf{1}} = \underline{\omega}_{00}\, e^{j(\Omega - v)t} \quad , \tag{11}$$

where $\underline{\omega}_{00}$ denotes ω at $t=0$ (the double zero is to avoid confusion with ω_o). Now the real and imaginary parts of (11) must separately be equal, and the left hand side (LHS) is easy to split into these parts as θ and φ are both real. Let us examine the class of solutions featuring precession without nutation. This requires the LHS of (11) to be purely imaginary throughout, and at once forces the exponent on the RHS to be identically zero. If that were not so, the real part of the RHS would fluctuate in time, contradicting the condition of no nutation. Since $v = I_o\omega_o / I$ and $\omega_o = \Omega + \dot{\varphi}$, we have the precession frequency from

$$\Omega - v = 0 \Rightarrow \dot{\varphi} = \frac{I_o\Omega}{I - I_o} \quad . \tag{12}$$

This result agrees with the standard FST results found in books, which one can verify using the formula for ω_o. This condition alone is clearly not sufficient to guarantee regular precession as we must also have $\underline{\omega}_{00}$ purely imaginary. This is reasonable as it means that an initial impulse in the φ direction is necessary to initiate nutation-free precession.

We now analyse a more general case. Suppose the initial condition features impulses in both θ and φ directions such that $\underline{\omega}_{00} = \omega_{01} + j\omega_{0j}$. Then applying (11) and separating real and imaginary parts yields

$$\dot{\theta} = \omega_{01}\cos(\Omega - v)t - \omega_{0j}\sin(\Omega - v)t \quad , \tag{13a}$$

$$\theta\dot{\varphi} = \omega_{01}\sin(\Omega - v)t + \omega_{0j}\cos(\Omega - v)t \quad . \tag{13b}$$

(13a) is integrated to yield

$$\theta = C + \frac{\omega_{01}}{\Omega - v}\sin(\Omega - v)t + \frac{\omega_{0j}}{\Omega - v}\cos(\Omega - v)t \quad . \tag{14}$$

Now we can see if C equals zero, then the RHS of (14) will exactly divide that of (13b) and we will be left with uniform precession at a frequency Ω-v. Now C depends on the initial conditions. However we note that the choice of the lab frame axes in this scenario is completely arbitrary as there is no force of any kind on the top, and *we can always choose the* x-y-z *system such that* C *evaluates to zero.* In Lagrangian terms we can say that θ is a cyclic

coordinate for the FST and can thus be chosen arbitrarily. A suitable choice of the axes thus reduces the solution to nutation, combined with uniform precession at frequency Ω-v, which on substituting the value of v evaluates to

$$\dot{\varphi} = \frac{I_o\Omega}{2I - I_o} \quad . \tag{15}$$

The argument going on here is rather subtle so we explain it once more. We note that in our formalism equations (1) to (9) are independent of the axes used since they involve vectors which are left unchanged by axes transformations. Their components of course change but they themselves do not. This property gives us the licence to choose our lab frame axes arbitrarily. The d, q and o axes are of course fixed (to within a constant ψ) but the x, y and z axes are not. The definitions of the Eulerian angles remain valid for any choice of the lab frame axes. Hence for a suitable choice of axes we can always view the top motion as a combination of steady precession and nutation *about those axes*. Let us see how to obtain the suitable lab frame axes. We fix the d-q-o axes so that ω_{01} and ω_{0j} also become fixed. Then the angle θ must be defined such that at t=0, it has the value $\dfrac{\omega_{0j}}{\Omega - v}$. This fixes the lab frame axes about which the motion has a simple description. The nutation is described by

$$\theta = \frac{2I - I_o}{I_o\Omega}\left(\omega_{01}\sin\frac{I_o\Omega t}{2I - I_o} + \omega_{0j}\cos\frac{I_o\Omega t}{2I - I_o}\right) \quad . \tag{16}$$

From these we recover the known result that the nutation frequency increases and the amplitude decreases as the top spins faster and faster.

We also see that the nutation frequency tends to blow up for a particular ratio between the two moments of inertia namely I_o:I=2:1 – we note that this is the largest possible value the ratio can have. To increase the ratio, we must have as little mass distribution along the o axis as possible and circular symmetry forces the shape of the top to be a solid of revolution. The two criteria indicate that the shape of top which maximises the ratio is the plane disk – then too the ratio of 2 is achieved if the pivot be the same as the centre of mass – not a realistic top at all. Any deviation from this ideal shape only tilts the balance in favour of I (which had better be so).

Our method has thus yielded the FST solutions and we now try other, more complicated cases. We note that the FST solutions are always relevant as they are in fact the homogeneous solutions of (5).

2. HEAVY SYMMETRIC TOP (HST)

We now take on the HST using our formalism. Fig. 1 shows that the torque of gravity acts along the a axis; in fact along the $-\hat{\mathbf{a}}$ direction as it is the cross product of the displacement ($\hat{\mathbf{c}}$) and the force ($\hat{\mathbf{b}}$). The magnitude of the torque is $Mgh\theta$ where h is the distance along the o axis from the pivot to the centre of mass of the top. The $\underline{\mathbf{\Gamma}}$ phasor is obtained by applying (6) on this torque, and substituting this into (5) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega} + jv\underline{\omega} = -\frac{Mgh\theta}{I}\underline{\mathbf{1}}\mathrm{e}^{-j\Omega t} \quad . \tag{17}$$

This equation of course has homogenous and particular solutions – the homogeneous ones are nothing but the FST solutions, (10)ff. and the particular solutions are the ones on which we will focus now. This solution is, using subscript p for particular, and assuming θ_p to vary slowly in comparison with Ωt:

$$\underline{\omega}_p = j\frac{Mgh\theta_p}{I(\Omega - v)}\underline{\mathbf{1}}\mathrm{e}^{-j\Omega t} \quad , \tag{18}$$

on which (9) is applied to yield

$$\dot{\theta}_p + j\theta_p\dot{\varphi}_p = j\frac{Mgh\theta_p}{I(\Omega - v)} \quad . \tag{19}$$

The RHS of the above is purely imaginary hence we at once have $\dot{\theta}_p = 0$ whereby θ_p becomes a constant. Hence our slowly varying θ assumption leads to no error after all. Equating the imaginary parts yields the precession frequency; the equation is implicit however as v depends on ω_o which itself has a contribution from $\dot{\varphi}$. Writing things explicitly we recover a quadratic for $\dot{\varphi}$:

$$\left(I_o - I\right)\dot{\varphi}^2 + I_o\Omega\dot{\varphi} + Mgh = 0 \quad , \tag{20}$$

which has the usual slow and fast precession frequency solutions.

As in the previous case, we have again derived the HST results with considerable ease. The treatment is not yet over though, as our formalism provides a direct link between the release conditions and the top's subsequent behaviour. This in fact is a topic which is often not discussed explicitly in books on mechanics – they give the dynamical equations but then do not mention what the top will actually do if say released from rest at an angle or released from vertical with an angular impulse. This aspect of the problem is ideally suited to our approach.

There are only three release parameters of a HST which make physical sense. They are the launching angle θ, and the launch time angular velocities in the θ and φ direction. φ is a cyclic or ignorable coordinate for the HST and we can choose our axes such that its initial value is zero. Let us say the initial launching conditions are θ_{00}, $\dot{\theta}_{00}$ and $\dot{\varphi}_{00}$. As before we have used the double nought subscript to avoid confusion with o axis components. Now $\left[\dot{\theta}_p\right]_{t=0} = 0$ and $\left[\dot{\varphi}_p\right]_{t=0} = \dot{\varphi}_{fast}$ or $\dot{\varphi}_{slow}$ depending on the precession mode which the top chooses to go into. Generally it is the slow mode which is preferred, although it is not relevant here. The difference between the initial values and the $t{=}0$ particular solutions gives the $t{=}0$ homogeneous solutions (subscript h). But we know that these evolve according to (11) :

$$\left(\dot{\theta}_h + j\theta_h\dot{\varphi}_h\right)\mathbf{\underline{1}} = \left(\left[\dot{\theta}_h\right]_{t=0} + j\theta_{00}\left[\dot{\varphi}_h\right]_{t=0}\right)\mathbf{\underline{1}}\,e^{j(\Omega-v)t} \quad . \tag{21}$$

And a closed form solution of (11) can always be obtained, if need be by reorienting the axes as discussed after (14). Of course, this time the *x-y-z* axes are not arbitrary as gravity makes the space anisotropic. However the axis reorientation is eminently permissible while treating the homogeneous torque-free solutions. Some messy axis transformations may be required to change from the reoriented axes to the original ones and thus obtain the precession and nutation in the gravity-aligned basis. But this procedure can be done in principle as well as in practice, and we have thus given a complete recipe for predicting the behaviour of a HST given its launch conditions.

One of the demerits of the standard formulation of the top problem is that the effects of friction at the pivot are neglected. As Goldstein comments, "the effects of friction....cannot be directly included in the Lagrangian framework". Nevertheless he also says that, for a practical top, "the nutation is damped out by friction at the pivot..." and "the top then appears to precess uniformly". This statement is in fact not intuitive as one may ask why the friction selectively damps out the nutation leaving the precession unharmed. We see that our approach is amenable to the insertion of a friction term and the analysis of this system is the focus of the next subsection.

3. DAMPED COMPENSATED (DC) TOP

A reasonable model of friction at the pivot features a torque proportional to the angular velocity, of course with a minus sign. The body frame angular velocity components are the natural choice as that is the only significant frame from the viewpoint of the top (the lab frame becomes special only if gravity is present but we assume the friction acts on the FST too). Hence we make the insertions of the form

$$\Gamma_d = -\gamma_1 \omega_d \quad , \tag{22a}$$

$$\Gamma_q = -\gamma_2 \omega_q \quad , \tag{22b}$$

$$\Gamma_o = -\gamma_3 \omega_o \quad , \tag{22c}$$

where the γ's (all greater than zero) denote damping coefficients which need not be equal on the three axes. But common sense says that they will be equal for the d and q axes and much less for the o axis. For instance, a crude gyroscope may use a ball bearing for mounting the rotor on its own axis but may use an ordinary (sliding) pivot to make contact with the ground. More sophisticated gyros will use a rotor powered by a motor which will produce a torque to counter the frictional effect and there will be no effective damping on the o axis. Damping on the o axis is anyway an undesirable phenomenon as it will be a hindrance in experimentation using the gyro. Hence we assume that $\gamma_1 = \gamma_2 = \gamma$ and $\gamma_3 = 0$. This last property is what we refer to by "compensation" in the title of this subsection – the o axis damping is compensated by an external motor. Accordingly we have the modified form of (5) :

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega} + (\gamma + j\nu)\underline{\omega} = \frac{1}{I}\underline{\Gamma} \quad . \tag{23}$$

This is the governing equation of the DCT. The homogeneous solution this time decays quickly to zero. The damped compensated FST (DCFST) will simply come to rest at some position spinning away without precessing or nuting, and this does not make for a worthwhile study. The damped compensated HST (DCHST) is more interesting and we start by writing its characteristic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\omega} + (\gamma + j\nu)\underline{\omega} = -\frac{Mgh\theta}{I}\underline{1}\mathrm{e}^{-j\Omega t} \quad . \tag{24}$$

Since the homogenous solutions are damped out we focus only on the particular integral, this time omitting the subscript p which we had used in Subsection II.2. Solving (24), applying (9) and separating real and imaginary gives the following equations :

$$\dot{\theta} = \frac{-\gamma Mgh}{I\left(\gamma^2 + (\Omega - \nu)^2\right)}\theta \quad , \tag{25a}$$

$$\dot{\varphi} = \frac{(\Omega - \nu)Mgh}{I\left(\gamma^2 + (\Omega - \nu)^2\right)} \quad . \tag{25b}$$

Like (19) the equation for the precession is implicit and must be unscrambled to form a simple cubic; however that is not really too relevant. The presence of a damping term suggests that a fast precession state will not be favoured. For slow precession we note that the term Ω-ν evaluates to $I_o\Omega/I$. Now (25a) yields that the polar angle starts out from its initial value and decays to zero with a certain time constant. Making the assumption of slow precession, this time constant evaluates to

$$\tau = \frac{I^2\gamma^2 + I_o^2\Omega^2}{\gamma IMgh} \quad . \tag{26}$$

We note that this is quite different from the time constant $1/\gamma$ with which the homogeneous solutions die out. In fact for a very fast top, the homogeneous solutions are killed off much quicker than the time it takes for the angle to vanish. As the nutation arises only from the homogeneous solution [(25a) does not describe nutation under any circumstances] we have mathematically explained Goldstein's comment that the nutation dies out and the top appears to process uniformly.

Another interesting feature of the solution is that the polar angle slowly but surely goes to zero. For very small θ, the precession is meaningless as it is scarcely visible. Hence the eventual state of the top is to spin uniformly

pointing steadily in the upward vertical direction. It is a classic display of levitation – the top simply stands in equilibrium at the position which gravity would have considered the most unstable, mocking gravity to bring it down if it can. The ease of obtaining this solution encourages us to apply the method to a still more complicated case where the pivot is given an acceleration in the horizontal plane. That is the topic of the next subsection.

4. DCHST WITH HORIZONTAL ACCELERATION

In this subsection we consider the DCHST where the pivot experiences a uniform horizontal acceleration (a in the $-\hat{\mathbf{x}}$ direction to be specific, not to be confused with the similar label of the axis, which does not appear in this subsection). As in the previous section we are interested in the long time state. Given the previous results, the problem is in fact a one-liner : the final state is no precession and no nutation, $\theta = \tan^{-1}(a/g) \approx a/g$ and $\varphi=\pi/2$. In the pivot frame the centre of mass (CM) of the top experiences a pseudo force $+Ma$ in the $\hat{\mathbf{x}}$ direction which is entirely equivalent to the gravitational force. Now the choice of the lab frame axes is arbitrary and we will recover the previous problem *in toto* if we align $-z$ with the direction of the net force on the CM. The final state of the top would be to point along this new $+z$ which in this case means vertically upward and in the direction of the acceleration. An easy check shows that the new z axis does indeed make the angles we claimed above with the original axes, and the problem is solved.

In this paper however we are trying to see the applicability of our method so we prove that we can derive this result in our original *x-y-z* basis. The pseudo force on the CM is $Ma\hat{\mathbf{x}}$ and the torque is obtained by $\mathbf{r} \times \mathbf{F}$ as usual. Now for the *d-q-o* components of torque we must expand $\mathbf{F}$ in the *d-q-o* basis and for this purpose we use the first column of the transformation matrix (7). Evaluating the cross product we have

$$\mathbf{\Gamma} = Mah\left[\left(\sin\varphi\cos\psi+\cos\varphi\sin\psi\right)\hat{\mathbf{d}}+\left(\cos\varphi\cos\psi-\sin\varphi\sin\psi\right)\hat{\mathbf{q}}\right] \quad . \tag{27}$$

Recognizing the sines and cosines of sums, and adding on the gravitational torque we construct the $\underline{\mathbf{\Gamma}}$ phasor and the governing equation for the top is

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\boldsymbol{\omega}}+\left(\gamma+jv\right)\underline{\boldsymbol{\omega}}=\frac{Mh}{I}\left(g\theta+jae^{-j\varphi}\right)\underline{\mathbf{1}}e^{-j\Omega t} \quad . \tag{28}$$

The problematic term here is the exp($-j\varphi$) on the RHS. But we can save a lot of messy algebra by recognizing that the slow precessional states are favoured for the damped top. Under this condition the time variation of the term involving φ will be much less than that of the term featuring Ωt. This is the same argument as the one we used to treat θ as a constant in (18); as it happened we accrued no error in that case. Here, the error will not fortuitously evaluate to zero but we can definitely integrate (28) at least for a few cycles of spin, assuming φ to be a constant. This simplified integral will not be an exact description of the motion but let us see how much information we can get out of it. The particular solution with this assumption is

$$\underline{\boldsymbol{\omega}}_p = \frac{-Mgh\theta+jMahe^{-j\varphi}}{I\left[\gamma-j\left(I_o\Omega/I\right)\right]}\underline{\mathbf{1}}e^{-j\Omega t} \quad , \tag{29}$$

where we have written the form of Ω-v appropriate for slow precession.

Using (9) and separating the real and imaginary parts yields

$$\dot{\theta} = \frac{1}{I\left(\gamma^2+(I_o/I)^2\Omega^2\right)}\left[-\gamma Mgh\theta+\gamma Mah\sin\varphi-\frac{I_o\Omega}{I}Mah\cos\varphi\right] \quad , \tag{30a}$$

$$\theta\dot{\varphi} = \frac{1}{I\left(\gamma^2+(I_o/I)^2\Omega^2\right)}\left[-\frac{I_o\Omega}{I}Mgh\theta+\frac{I_o\Omega}{I}Mah\sin\varphi+\gamma Mah\cos\varphi\right] \quad . \tag{30b}$$

These equations are difficult to solve in detail and the easiest is to perform at this stage the axis rotation which we could have performed right at the beginning. The fixed point of the dynamics is evident though on inspection :

$$\varphi = \pi/2; \theta = a/g; \dot{\theta} = \dot{\varphi} = 0 \quad , \tag{31}$$

which we have already shown is the true fixed point of the system.

This at last completes the set of examples which we had wanted to demonstrate using our formalism. We close with a brief section where we summarise the main ideas and give some details regarding experimental visualization of the various conclusions we have drawn.

RECAPITULATION AND CONCLUSION

The main point of this paper has been the proposal of a novel mathematical formalism for handling problems on rigid body rotation. We have selected the body frame as the reference frame of choice and used Euler's equations (3) in it. The assumptions of symmetry and of no torque about the figure axis have resulted in the space phasor method being applicable on (3) to produce (5) which is one of our central equations. The conversion from the abstract body frame angular velocities to the concrete, observable ones has been done using axis transformations (7), resulting in the second key equation (9). Until this point (i.e. in Sec. I) we have made no simplifications but merely done a mathematical reformulation of the top problem. The usual system of two second order differential equations with somewhat messy non-linear terms has been converted to a system of two nested complex equations of which the first one is linear. We say "nested" as (5) is like the outer equation and (9) the inner equation – the solution of (5) thus gives rise to yet another differential equation. The mathematical structure might also result in an advantage in obtaining numerical solutions of the top dynamics. The computer does not have to solve multiple equations simultaneously as in the traditional formalism, but can do so serially.

The first application of the formalism is to the free symmetric top where the body frame analysis mirrors the standard treatment. The conversion to lab frame angular velocities has also been shown. The importance of this section is primarily because the free top solutions are in fact the homogeneous solutions of the heavy symmetric top. This homogeneous-particular approach is a direct method of determining the relation between the launch conditions of the heavy top and its subsequent motion. The particular solutions of (9) for the heavy top are given in (19); they admit uniform precession at a choice of two different frequencies, and no nutation. The $t=0$ homogeneous solutions are obtained from the initial conditions and their evolution is governed by (21), whose solutions allow for nutation.

Verification of the predictions of (19) and (21) needs a system where frictional effect is very low. One such experimental setup is a variation of the apparatus mentioned in Kleppner [9]. Our pivot consists of a large, light spherical ball which is kept in a closely fitting cup. Now the cup is connected to a powerful air blower so that the ball does not actually make contact with the cup but rides on an air cushion a few millimetres above it. This top can be subjected to various initial conditions and the agreement with our predictions can be measured.

For the damped compensated top we see that the homogeneous solutions which include fast precession and nutation decay exponentially in time. The particular solution features uniform precession at a choice of three frequencies but because of the damping term, the slow precession is expected to be prevalent. The polar angle also decreases exponentially in time and eventually the top becomes vertical; the time constant of this decrease (26) is however not equal to that of the homogeneous solutions. When the top is subjected to a horizontal acceleration in addition to gravity, it eventually points opposite to the direction of the net force plus pseudo force.

For experiments to verify these results, the primary feature to be careful of is that the viscous damping of our model is different from the static friction experienced at a practical rough pivot. An accurate result will be obtained if static friction is minimized and a damping is externally introduced. One way is to use a smooth ball and socket joint as the pivot and immerse the surrounding area in fluid. A still better implementation would be to construct the Kleppner top above with the ball made of a light metal like aluminium. A magnetic field can be set up in the region so that Lenz's law serves to retard the pivot's motion. Another subtlety which must be kept in mind is that the motor driving the top must produce a constant torque output to counter the frictional effect of the

pivot. It should not tend towards a constant speed mode of operation, which will result in unwanted o axis torques. For an uncontrolled motor, the series field DC motor in the flux weakening regime is the best choice as its torque speed curve is nearly flat at high speeds. If motor control algorithms are employed, the induction motor with direct torque control will be the ideal candidate.

*   *   *   *   *

ACKNOWLEDGEMENT

REFERENCES

[1] H. Goldstein, C. P. Poole and J. L. Safko, "*Classical Mechanics*," 3rd ed. Pearson Education, Upper Saddle River, New Jersey, USA (2002)

[2] L. D. Landau and E. M. Lifshitz, "*Mechanics*," Pergammon, New York, USA (1976)

[3] D. Morin, "*Introduction to Classical Mechanics*," Cambridge University Press, Cambridge UK (2007)

[4] J. Peraire and S. Windall, "3D rigid body dynamics : tops and gyroscopes," MIT OpenCourseWare **16.07** Dynamics (2008)

[5] K. Schonhammer, "Elementary theoretical description of the heavy symmetric top," Am. J. Phys. **66** (11), 1003-1007 (1998)

[6] A. Sommerfeld, "*Mechanics*," Academic Press (1956)

[7] R. Krishnan, "*Electric Motor Drives – Modeling, Analysis and Control*," PHI Learning Private Ltd. New Delhi (2010)

[8] S. Bhattacharjee, "The electrodynamics of the induction motor," Proc. Prog. in Electomagnetics. Res. Symp. (PIERS) in Moscow, 482-486 (2012)

[9] D. Kleppner and R. J. Kolenkow, "*An Introduction to Mechanics*," Cambridge University Press, Cambridge UK (2010)